The Euclidean Algorithm Through Squares: Area, Perimeter, and Continued Fractions

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Abstract

Starting with a rectangle, we repeatedly remove the largest square whose side is the shorter side of the current rectangle. We characterize finite termination, recast the classical Calkin–Wilf and Stern–Brocot level structure as a count of aspect ratios requiring a prescribed number of removals, and obtain exact formulas for the total area and the sum of the side lengths of the squares. The arguments are elementary and connect the geometric Euclidean algorithm with continued fractions and rational-number trees.

1 The construction and termination

Let the initial rectangle have side lengths $0 < a \le b$. At each stage, if the current rectangle has side lengths $0 < x \le y$, remove a square of side x and continue with the remaining rectangle of sides x and $y - x$. If the remaining rectangle is a square, we stop; for the area and side-length sums below, we will also count that *final* square. When $a = b$ initially, the rectangle is already a square and no removal is needed.

This is the subtractive geometric Euclidean algorithm; see, for example, [1, 2, 3, 4, 5, 12].

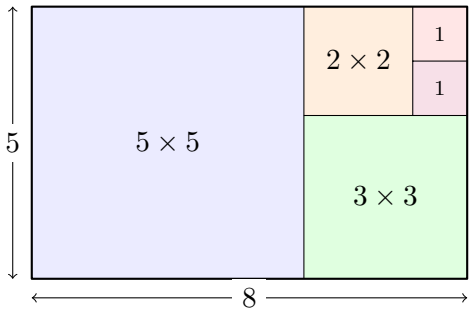


Figure 1: The square decomposition of a 5×8 rectangle. The successive squares have sides $5, 3, 2, 1, 1$; the last 1×1 square is the final square.

Theorem 1 (Termination)**.** *The construction reaches a square after finitely many removals if and only if $b/a \in \mathbb{Q}$. In that case, if $b/a = m/n$ in lowest terms and $a = nd$, $b = md$, the final square has side $d = \gcd(a, b)$ in the sense that d is the largest length for which both a and b are integer multiples of d.*

Proof. If $b/a = m/n$ in lowest terms, write $a = nd$ and $b = md$ for positive integers m, n and $d > 0$. Each removal replaces a pair of positive integer multiples of d by a pair with smaller sum, until

the two multiples are equal. Conversely, each subtraction preserves the additive group generated by the two side lengths. If the final square has side d, reversing the subtractions expresses both original side lengths as positive integer multiples of d. Hence b/a is rational.

For the last assertion, the additive group generated by the two side lengths is $d\mathbb{Z}$ at the start, because $\gcd(m, n) = 1$, and it is unchanged by every subtraction. A square of side d' generates $d'\mathbb{Z}$, so $d' = d$. □

Lemma 2 (Shrinking)**.** *If b/a is irrational, then after M removals both sides of the remaining rectangle tend to 0 as $M \to \infty$; in particular its area and perimeter tend to 0.*

Proof. Put $r_0 = b$, $r_1 = a$ and let $r_{i+1} = r_{i-1} - q_i r_i$ with $q_i = \lfloor r_{i-1}/r_i \rfloor$ be the Euclidean remainders. If some r_{i+1} were 0, then r_{i-1}/r_i and hence b/a would be rational, so all $r_i > 0$. Since $r_{i+1} < r_i$ and $r_i + r_{i+1} \le r_{i-1}$, we have $2r_{i+1} < r_{i-1}$, so $r_i \to 0$. While the q_i squares of side r_i are being removed, the rectangle has sides r_i and $r_{i-1} - jr_i$, both at most r_{i-1}. Each block is finite, so $i \to \infty$ as $M \to \infty$, and both sides tend to 0. □

1.1 Exactly N removals

We count a *removal* as the subtraction of one square, and stop counting as soon as the remaining rectangle is square. Thus a square initially requires $N = 0$ removals. For $b > a$, write the canonical finite simple continued fraction

$$\frac{b}{a} = [q_0; q_1, \ldots, q_k], \qquad q_0 \ge 1, \quad q_j \ge 1 \ (j \ge 1), \quad q_k \ge 2 \ (k \ge 1).$$

When $k = 0$, $q_0 \ge 2$. The quotients count blocks of equal-sized squares, and the last square is left in place. Consequently,

$$N = q_0 + q_1 + \cdots + q_k - 1. \tag{1}$$

For example, $8/3 = [2; 1, 2]$ requires $2 + 1 + 2 - 1 = 4$ removals.

Theorem 3 (Counting aspect ratios)**.** *For every integer $N \ge 1$, there are exactly 2^{N-1} distinct ratios $b/a > 1$ for which the process reaches a square after exactly N removals. Their minimum and maximum are*

$$\min \frac{b}{a} = \frac{N+1}{N}, \qquad \max \frac{b}{a} = N + 1.$$

Proof. A ratio $b/a > 1$ that reaches a square has a unique canonical expansion (last quotient at least 2 when $k \ge 1$), and by (1) the number of removals is $N = q_0 + \cdots + q_k - 1$. Put $p_j = q_j$ for $j < k$ and $p_k = q_k - 1$. These are positive integers (for $k = 0$, $p_0 = q_0 - 1 \ge 1$) with $p_0 + \cdots + p_k = N$. Conversely, every composition $(p_0, \ldots, p_k)$ of N gives the canonical expansion $[p_0; \ldots, p_{k-1}, p_k + 1]$, since $p_k + 1 \ge 2$. Distinct canonical expansions give distinct rationals, and N has 2^{N-1} compositions, so there are exactly 2^{N-1} ratios.

For the bounds we use the following fact: if $p_0, \ldots, p_r \ge 1$, then $[p_0; p_1, \ldots, p_r] \le p_0 + \cdots + p_r$. Indeed, for $r = 0$ this is an equality, and for $r \ge 1$ the value is $p_0 + 1/y \le p_0 + 1 \le p_0 + p_1 + \cdots + p_r$, where $y = [p_1; \ldots, p_r] \ge 1$.

Upper bound: $b/a \le q_0 + \cdots + q_k = N + 1$. For $k = 0$ we have $b/a = q_0 = N + 1$. For $k \ge 1$ the tail satisfies $q_1 + \cdots + q_k \ge q_k \ge 2 > 1$, so the inequality is strict. Hence the maximum $N + 1$ is attained only at $[N + 1]$.

Lower bound: if $q_0 \ge 2$, then $b/a \ge 2 \ge 1 + 1/N$. If $q_0 = 1$, write $b/a = 1 + 1/y$ with $y = [q_1; \ldots, q_k] \le q_1 + \cdots + q_k = N$ by the fact above. Hence $b/a \ge 1 + 1/N$, with equality for $[1; N]$ when $N \ge 2$. For $N = 1$ the only ratio is $2 = (N + 1)/N$. □

Remark 4 (The tree structure)**.** *The ratios with exactly N removals form the levels of a binary tree. From a reduced fraction $m/n > 1$ with N removals arise the two fractions*

$$\frac{m+n}{n} \quad \textit{and} \quad \frac{m+n}{m}$$

with $N+1$ removals: removing the first square from either one leaves a rectangle of ratio m/n. Conversely, the parent of a ratio is recovered by subtracting the smaller of numerator and denominator from the larger. The root at $N = 1$ is $2/1$.

This is the Calkin–Wilf recursion [9] read modulo reciprocals. Equivalently, the ratios above 1 with N removals are the numbers above 1 at depth N of the Stern–Brocot tree, whose depth-d numbers are those with partial quotients summing to $d+1$ [6, 7, 8, 10]. Theorem 3 is therefore a restatement of the level structure of these trees in the language of removals; see also [11].

The first few sets, written in increasing order, are

$$\begin{aligned}
N = 1 &: \quad \{2\}, \\
N = 2 &: \quad \{3/2, 3\}, \\
N = 3 &: \quad \{4/3, 5/3, 5/2, 4\}, \\
N = 4 &: \quad \{5/4, 7/5, 8/5, 7/4, 7/3, 8/3, 7/2, 5\}.
\end{aligned}$$

2 Total area of the squares

Let s_j be the side lengths of the removed squares, and include the final square when the process terminates.

Theorem 5 (Area identity)**.** *For every rectangle, whether the construction is finite or infinite,*

$$\sum_j s_j^2 = ab.$$

Proof. After M square removals, let the remaining rectangle have area A_M. Then

$$ab = \sum_{j=1}^{M} s_j^2 + A_M.$$

In the rational case, counting the final square gives $\sum_j s_j^2 = ab$. In the irrational case, Lemma 2 gives $A_M \to 0$. Hence again $\sum_j s_j^2 = ab$. The claimed identity follows. □

Remark 6. *In the infinite case, at every finite stage a nonempty rectangle remains. Nevertheless, the area of the remainder tends to zero, so the sum of the square areas equals ab. The identity concerns area, not finite completion of the construction.*

3 Sum of the side lengths

The sum of the side lengths has a direct explanation in terms of perimeter.

Proposition 7 (Perimeter decrement)**.** *Removing a square of side s from a rectangle decreases the perimeter of the remaining rectangle by $2s$.*

Proof. If the sides before removal are $x \le y$, the perimeter changes from $2(x+y)$ to $2y$, a decrease of $2x = 2s$. □

Theorem 8 (Sum of the side lengths)**.** *If b/a is irrational, then*

$$\sum_{j=1}^{\infty} s_j = a + b.$$

If $b/a = m/n$ in lowest terms, where $a = nd$ and $b = md$, then the sum of the side lengths of all squares, including the final square of side d, is

$$\sum_{j} s_j = a + b - d.$$

If the final square is not counted, the sum is $a + b - 2d$.

Proof. Telescoping the perimeter decrements gives, after M removals,

$$2\sum_{j=1}^{M} s_j = 2(a+b) - P_M,$$

where P_M is the perimeter of the remaining rectangle. For irrational b/a, Lemma 2 gives $P_M \to 0$, proving the first formula. In the rational case, stop at the final square, whose side is d by Theorem 1; its perimeter is $4d$. The sum of the side lengths of the removed squares is therefore $a + b - 2d$. Counting the final square adds d. □

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