

SAMPLE — AP Calculus AB Weekly Math Study Plan

Applications of Derivatives

A demonstration of the type of personalized study package prepared from a student's weekly assignments.

Homework Analysis

Suppose this week's assigned homework contains about 35–40 questions from an AP Calculus AB unit on applications of derivatives. Although the assignment contains many individual questions, most of them rely on a small number of recurring mathematical skills.

For this sample, the homework has been organized into the following six skills:

1. Find critical numbers and use the sign of f' to determine where a function increases or decreases.
2. Use the First Derivative Test to identify local maxima and minima.
3. Use f'' to determine concavity and locate inflection points.
4. Find absolute extrema on a closed interval.
5. Read derivative information from a graph or sign chart.
6. Set up and solve an optimization problem.

How to use this plan. Work through each skill below. After studying the example, try the short practice problem without looking back. When all six skills are comfortable, complete the Mastery Check near the end.

Skill 1: Critical Numbers and Increasing/Decreasing Intervals

A **critical number** of f is a number c in the domain of f for which

$$f'(c) = 0$$

or $f'(c)$ does not exist.

Critical numbers divide the domain into intervals. On each interval, test the sign of f' :

$$f'(x) > 0 \implies f \text{ is increasing}, \quad f'(x) < 0 \implies f \text{ is decreasing}.$$

Worked example. Let

$$f(x) = x^3 - 3x^2 - 9x + 5.$$

Then

$$f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1).$$

Thus the critical numbers are $x = -1$ and $x = 3$.

Testing the three intervals gives

x	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
$f'(x)$	+	−	+

Therefore f is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing on $(-1, 3)$.

Targeted practice. For

$$g(x) = x^3 - 6x^2 + 9x + 1,$$

find the critical numbers and the intervals on which g is increasing and decreasing.

Skill 2: Local Extrema and the First Derivative Test

The First Derivative Test looks at how the sign of f' changes at a critical number.

- If f' changes from positive to negative, f has a local maximum.
- If f' changes from negative to positive, f has a local minimum.
- If the sign does not change, there is no local extremum there.

Worked example. Return to

$$f(x) = x^3 - 3x^2 - 9x + 5.$$

From the sign chart above, f' changes $+$ $\rightarrow$ $-$ at $x = -1$, so f has a local maximum there. It changes $- \rightarrow +$ at $x = 3$, so f has a local minimum there.

The corresponding function values are

$$f(-1) = 10, \quad f(3) = -22.$$

Thus the local maximum point is $(-1, 10)$ and the local minimum point is $(3, -22)$.

Targeted practice. Use the First Derivative Test on the function g from Skill 1. State each local extremum as a point.

Skill 3: Concavity and Inflection Points

Concavity is determined by the second derivative:

$$f''(x) > 0 \implies f \text{ is concave up,}$$

$$f''(x) < 0 \implies f \text{ is concave down.}$$

An inflection point occurs where the graph changes concavity. Solving $f''(x) = 0$ gives candidates, but the concavity must actually change.

Worked example. Let

$$h(x) = x^4 - 4x^3.$$

Then

$$h'(x) = 4x^3 - 12x^2, \quad h''(x) = 12x^2 - 24x = 12x(x - 2).$$

The candidates are $x = 0$ and $x = 2$. A sign test gives

x	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$h''(x)$	+	-	+

so h is concave up on $(-\infty, 0)$ and $(2, \infty)$ and concave down on $(0, 2)$.

Both candidates are genuine inflection points:

$$h(0) = 0, \quad h(2) = -16.$$

Thus the inflection points are $(0, 0)$ and $(2, -16)$.

Targeted practice. Determine the intervals of concavity and all inflection points of

$$p(x) = x^4 - 8x^2.$$

Skill 4: Absolute Extrema on a Closed Interval

For a continuous function on $[a, b]$, the absolute maximum and minimum can occur at critical numbers *or* at *endpoints*.

Use the Closed Interval Method:

1. Find critical numbers inside (a, b) .
2. Evaluate f at those critical numbers.
3. Evaluate $f(a)$ and $f(b)$.
4. Compare all values.

Worked example. Find the absolute extrema of

$$q(x) = x^3 - 3x, \quad -2 \leq x \leq 3.$$

Since

$$q'(x) = 3x^2 - 3 = 3(x - 1)(x + 1),$$

the critical numbers are $x = -1$ and $x = 1$.

Evaluate:

$$q(-2) = -2, \quad q(-1) = 2, \quad q(1) = -2, \quad q(3) = 18.$$

Therefore the absolute maximum is 18 at $x = 3$, and the absolute minimum is -2 , attained at both $x = -2$ and $x = 1$.

Targeted practice. Find the absolute maximum and minimum of

$$r(x) = x^3 - 12x, \quad -3 \leq x \leq 4.$$

Skill 5: Reading Information from the Derivative

Sometimes AP Calculus gives information about f' rather than a formula for f .

Suppose the sign of f' is

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, \infty)$
$f'(x)$	$-$	0	$+$	0	$-$

Then:

- f decreases on $(-\infty, -2)$,

- f increases on $(-2, 1)$,
- f decreases on $(1, \infty)$,
- f has a local minimum at $x = -2$,
- f has a local maximum at $x = 1$.

The key idea is that the *sign* of f' tells you the direction in which f moves.

Targeted practice. Suppose f' is positive on $(-4, 0)$, negative on $(0, 5)$, and positive on $(5, \infty)$, with $f'(0) = f'(5) = 0$. State where f increases and decreases and identify the local extrema.

Skill 6: Optimization

Optimization problems convert a verbal situation into a one-variable function that must be maximized or minimized.

A reliable procedure is:

1. Identify the quantity to optimize.
2. Write a constraint.
3. Use the constraint to express the objective in one variable.
4. Differentiate and find critical numbers.
5. Check that the critical number gives the required maximum or minimum.

Worked example. A rectangle has perimeter 40 meters. Find the dimensions that maximize its area.

Let the sides be x and y . The constraint is

$$2x + 2y = 40,$$

so

$$y = 20 - x.$$

The area is therefore

$$A(x) = x(20 - x) = 20x - x^2.$$

Differentiate:

$$A'(x) = 20 - 2x.$$

Setting $A'(x) = 0$ gives $x = 10$. Then $y = 10$.

Also,

$$A''(x) = -2 < 0,$$

so the critical point gives a maximum.

Thus the rectangle of maximum area is a 10 m by 10 m square.

Targeted practice. A farmer has 120 meters of fencing to enclose a rectangular field along a straight river. No fence is needed along the river. Find the dimensions that maximize the enclosed area.

Mastery Check

Try these without looking back at the worked examples.

1. Let

$$f(x) = x^3 - 12x + 1.$$

Find the critical numbers, determine where f is increasing and decreasing, and classify each local extremum.

2. Let

$$g(x) = x^4 - 6x^2.$$

Determine the intervals on which g is concave up and concave down, and find all inflection points.

3. Find the absolute maximum and absolute minimum of

$$h(x) = x^3 - 3x^2 + 1$$

on the interval $[-1, 3]$.

4. Suppose $F'(x)$ is negative on $(-\infty, -1)$, positive on $(-1, 4)$, and negative on $(4, \infty)$. Identify the local extrema of F .
5. An open-top box is made from a 20 inch by 20 inch square sheet of cardboard by cutting equal squares of side length x from the four corners and folding up the sides.
- (a) Write the volume $V(x)$.
- (b) State the physically meaningful domain.
- (c) Find the value of x that maximizes the volume.

Before You Return to Your School Homework

You are ready to return to the assigned homework if you can:

- find critical numbers without confusing them with extrema;
- use signs of f' to determine increasing/decreasing behavior;
- use signs of f'' to determine concavity;
- remember to test endpoints in closed-interval absolute-extrema problems;
- interpret a derivative sign chart without needing a formula for the original function; and
- turn an optimization constraint into a one-variable objective function.

If one of these still feels uncertain, review that skill before continuing with the assigned homework.

PARENT ANSWER KEY

AP Calculus AB — Applications of Derivatives

Please keep this section separate from the student study materials.

This section is intended for the parent or guardian. It contains the answers to the Targeted Practice problems and the Mastery Check so you can check the student's understanding and identify any topics that may need additional review.

Part I: Targeted Practice Answers

Skill 1.

$$g'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3).$$

Critical numbers: $x = 1, 3$. Increasing on $(-\infty, 1)$ and $(3, \infty)$; decreasing on $(1, 3)$.

Skill 2. At $x = 1$, g' changes $+$ $\rightarrow$ $-$, so there is a local maximum. At $x = 3$, g' changes $-$ $\rightarrow$ $+$, so there is a local minimum.

$$g(1) = 5, \quad g(3) = 1.$$

Local maximum: $(1, 5)$. Local minimum: $(3, 1)$.

Skill 3.

$$p'(x) = 4x^3 - 16x, \quad p''(x) = 12x^2 - 16 = 4(3x^2 - 4).$$

Thus

$$x = \pm \frac{2}{\sqrt{3}}.$$

The function is concave up on

$$\left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, \infty\right)$$

and concave down on

$$\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right).$$

Since

$$p\left(\pm \frac{2}{\sqrt{3}}\right) = \frac{16}{9} - \frac{32}{3} = -\frac{80}{9},$$

the inflection points are

$$\left(-\frac{2}{\sqrt{3}}, -\frac{80}{9}\right), \quad \left(\frac{2}{\sqrt{3}}, -\frac{80}{9}\right).$$

Skill 4.

$$r'(x) = 3x^2 - 12 = 3(x-2)(x+2).$$

Check $x = -3, -2, 2, 4$:

$$r(-3) = 9, \quad r(-2) = 16, \quad r(2) = -16, \quad r(4) = 16.$$

Absolute maximum: 16, attained at $x = -2$ and $x = 4$. Absolute minimum: -16 at $x = 2$.

Skill 5. Increasing on $(-4, 0)$ and $(5, \infty)$; decreasing on $(0, 5)$. Local maximum at $x = 0$; local minimum at $x = 5$.

Skill 6. Let x be the side perpendicular to the river and y the side parallel to it. Then

$$2x + y = 120, \quad y = 120 - 2x.$$

Thus

$$A(x) = x(120 - 2x) = 120x - 2x^2,$$

so

$$A'(x) = 120 - 4x = 0$$

gives $x = 30$. Then $y = 60$. The maximizing dimensions are 30 m by 60 m.

Part II: Mastery Check Answers

1.

$$f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2).$$

Critical numbers: $x = -2, 2$. Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 2)$. Local maximum at

$$(-2, 17),$$

and local minimum at

$$(2, -15).$$

2.

$$g''(x) = 12x^2 - 12 = 12(x - 1)(x + 1).$$

Concave up on $(-\infty, -1) \cup (1, \infty)$ and concave down on $(-1, 1)$. Since

$$g(\pm 1) = -5,$$

the inflection points are $(-1, -5)$ and $(1, -5)$.

3.

$$h'(x) = 3x^2 - 6x = 3x(x - 2).$$

Check $x = -1, 0, 2, 3$:

$$h(-1) = -3, \quad h(0) = 1, \quad h(2) = -3, \quad h(3) = 1.$$

Absolute maximum: 1, attained at $x = 0$ and $x = 3$. Absolute minimum: -3 , attained at $x = -1$ and $x = 2$.

4. At $x = -1$, F' changes $- \rightarrow +$, so F has a local minimum. At $x = 4$, F' changes $+ \rightarrow -$, so F has a local maximum.

5. After cutting out squares of side x , the base dimensions are $20 - 2x$ by $20 - 2x$ and the height is x . Therefore

$$V(x) = x(20 - 2x)^2, \quad 0 < x < 10.$$

Differentiate:

$$V'(x) = (20 - 2x)(20 - 6x).$$

The interior critical number is

$$x = \frac{10}{3}.$$

The volume approaches 0 at both ends of the physical interval, so the maximum occurs at

$$\boxed{x = \frac{10}{3} \text{ inches}}.$$

Using the Results

If the student misses a problem, use the skill identified by that problem to decide what to review before returning to the school assignment. The goal is not simply to count correct answers, but to identify which mathematical ideas are secure and which need another look.